Linear Algebra and its Applications 434 (2011) 319–326

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Star partial order on $B(H)$ ☆

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ARTICLE INFO

Article history:
Received 12 March 2010
Accepted 18 August 2010
Available online 17 September 2010

Submitted by L. Rodman

AMS classification:
06A06
15A03
47B99

Keywords:
Minus partial order
Star partial order

ABSTRACT

Let H be an infinite-dimensional complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. In the paper the equivalent definition of the star partial order on $B(H)$, using self-adjoint idempotent operators, is introduced. Also some properties of the generalized concept of order relations on $B(H)$, defined with the help of idempotent operators, are investigated.

1. Introduction

Let M_n be the algebra of all $n \times n$ complex matrices, let H be an infinite-dimensional complex Hilbert space, and let $B(H)$ be the algebra of all bounded linear operators on H. On M_n a lot of partial orders and their properties, which can not be fully generalized to $B(H)$, were studied. One of such orders is the star partial order, which was defined by Drazin [2] as

$$A \underset{*}{\leqslant} B \quad \text{if and only if} \quad A^*A = A^*B \quad \text{and} \quad AA^* = BA^*, \tag{1}$$

A, B from M_n or $B(H)$, A^* the adjoint of A, and was characterized on M_n as

$$A \underset{*}{\leqslant} B \quad \text{if and only if} \quad A^\dagger A = A^\dagger B \quad \text{and} \quad AA^\dagger = BA^\dagger, \tag{2}$$

$A, B \in M_n$.

☆ Supported in part by a grant from the Ministry of Higher Education, Science and Technology, Slovenia.

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doi:10.1016/j.laa.2010.08.023

The superscript $A^\dagger$ here denotes the Moore–Penrose inverse of $A \in M_n$ which is the unique solution $G \in M_n$ of the following simultaneous matrix equations

(i) $AGA = A$,
(ii) $GAG = G$,
(iii) $(AG)^* = AG$,
(iv) $(GA)^* = GA$.

These four equations are often called the Penrose equations. A generalized inner inverse or a g-inverse A^- of A is a solution of the first Penrose equation while the reflexive g-inverse A_r^- satisfies the first and the second Penrose equation. Hartwig showed in [4] that one may replace the conjugate transpose A^* in (1) or the Moore–Penrose inverse $A^\dagger$ in (2) by a reflexive g-inverse A_r^- and still keep a partial order on M_n, so

$$A \ll B \quad \text{if and only if} \quad A_r^- A = A_r^- B \quad \text{and} \quad AA_r^- = BA_r^- .$$

It was also shown in [4] that this order is equivalent to the rank substractivity order (see also[5]) which is defined by

$$A \ll B \quad \text{if and only if} \quad \operatorname{rank}(B - A) = \operatorname{rank} B - \operatorname{rank} A.$$

Later it was observed by Hartwig that there exists another equivalent definition of the rank substractivity order, namely

$$A \ll B \quad \text{if and only if} \quad A^- A = A^- B \quad \text{and} \quad AA^- = BA^- \tag{3}$$

for some choice of a generalized inner inverse A^-. The partial order $\ll$ is thus usually called the minus partial order.

Note that in the definition of the minus partial order on M_n ranks or generalized inverses of matrices were used. Clearly definition with ranks can not be fully generalized to operator algebras on infinite-dimensional Hilbert space. And it is similar also with generalized inverses. Recall that for an arbitrary $A \in M_n$ there exist invertible matrices S and T such that $SAT = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ where I is the $r \times r$ identity matrix and $r = \operatorname{rank} A$. Let us denote by $\mathcal{G}(A)$ the set of all inner generalized inverses of A. One may check that then $\mathcal{G}(A) = T\mathcal{A}S$ where $\mathcal{A} \subseteq M_n$ is the set of all matrices of the form $\begin{bmatrix} I & * \\ * & * \end{bmatrix}$. Here, $*'s$ stand for arbitrary matrices of the appropriate sizes. For $A, B \in M_n$ it follows that

$$A \ll B \text{ if and only if } \mathcal{G}(B) \subseteq \mathcal{G}(A).$$

Recently Šemrl [8], while extending the minus partial order from M_n to $B(H)$, noted that one might do this extension by comparing the sets of generalized inner inverses of operators from $B(H)$. However he did not find this approach satisfactory. Namely, $A \in B(H)$ has a generalized inner inverse if and only if its image is closed (see for example[7]). Since Šemrl did not want to restrict his attention only to closed range operators, he found a new approach how to extend the minus partial order from M_n to $B(H)$. He took the standard definition (3) of the minus partial order on M_n, then found an appropriate equivalent definition, and then extended it to $B(H)$. More precisely, he proved that for $A, B \in M_n$ we have $A \ll B$ if and only if there exist idempotent matrices $P, Q \in M_n$ such that $\operatorname{Im} P = \operatorname{Im} A$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$ and $AQ = BQ$. Of course, the image of a bounded linear operator is closed, so Šemrl extended the concept of the minus partial order from M_n to $B(H)$ by replacing $\operatorname{Im} A$ in the first of the four equations by its closure. Šemrl's definition of the minus partial order on $B(H)$ is therefore as follows.

Definition 1. Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A, B \in B(H)$ we write $A \ll B$ if and only if there exist idempotent operators $P, Q \in B(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
(ii) $\operatorname{Ker} A = \operatorname{Ker} Q$,

(iii) $PA = PB$, and
(iv) $AQ = BQ$.

The order $\ll$ is called the minus partial order on $B(H)$.

Šemrl proved in [8] that $\ll$ is indeed a partial order.

Our goal is to extend Šemrl's approach presented in Definition 1 also to the star partial order on the algebra $B(H)$. Since the star partial order, contrary to the minus partial order, is defined on even more general settings than $B(H)$, see [2], it would suffice to show that on $B(H)$ definition of the star partial order with selfadjoint idempotents (see Definition 2) is equivalent to the usual definition of the star partial order (see equation (1)). However, since the proof that the relation (1) is indeed the star partial order on $B(H)$ was only announced [2,3], we will start with the proof that Definition 2 really defines a partial order and then proceed with the proof of equivalence of the mentioned partial orders. We believe that the former proof gives some new insight in the structure of the partial orders on $B(H)$.

2. An equivalent definition of the star partial order on $B(H)$

It is known that for $A, B \in M_n$ from $A \underset{*}{\leqslant} B$ it follows that $A \ll B$. Generally, the reverse implication does not hold. So, when defining the star partial order on $B(H)$ through Šemrl's approach, we should add some conditions beside the four Šemrl's equations. We found out that on M_n the usual definition of the star partial order (1) is equivalent to Definition 1 with additional requirements that $P = P^*$ and $Q = Q^*$, that is $A \underset{*}{\leqslant} B, A, B \in M_n$, if and only if there exist idempotent matrices $P, Q \in M_n$ such that $PA = PB, AQ = BQ, \operatorname{Im} P = \operatorname{Im} A, \operatorname{Ker} Q = \operatorname{Ker} A, P = P^*$, and $Q = Q^*$. We will skip the proof, since later on we will present the proof of equivalence of definitions on $B(H)$, H an arbitrary Hilbert space.

Therefore we generalize the definition of the star partial order to $B(H)$ using Šemrl's approach in the following way.

Definition 2. Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A, B \in B(H)$ we write $A \underset{*}{\leqslant} B$ if and only if there exist selfadjoint idempotent operators $P, Q \in B(H)$ such that

(i) $\operatorname{Im} P = \overline{\operatorname{Im} A}$,
(ii) $\operatorname{Ker} A = \operatorname{Ker} Q$,
(iii) $PA = PB$,
(iv) $AQ = BQ$.

The order $\underset{*}{\leqslant}$ is called the star partial order on $B(H)$.

Our next goal is to prove that $\underset{*}{\leqslant}$ is indeed a partial order on $B(H)$. Let C, D be subsets of a Hilbert space H such that $\langle x, y\rangle = 0$ for every $x \in C$ and for every $y \in D$. We say that C and D are orthogonal and denote $C \perp D$. First let us prove the following lemma, which is similar to Theorem 2 from [8].

Lemma 3. *Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. If $A, B \in B(H)$, then the following statements are equivalent.*

(*a*) $A \underset{*}{\leqslant} B$.

(*b*) *There exist closed subspaces H_1, H_2 of H such that $A, B : H_1 \oplus H_1^{\perp} \to H_2 \oplus H_2^{\perp}$ have matrix representations*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},$$

where $A_1 : H_1 \to H_2$ *and* $B_1 : H_1^{\perp} \to H_2^{\perp}$ *are bounded linear operators and* A_1 *is injective with* $\overline{\operatorname{Im} A_1} = H_2$.

(*c*) $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(B-A)}$ *and* $\overline{\operatorname{Im} A^*} \perp \overline{\operatorname{Im}(B^*-A^*)}$.

Proof. First, let us prove the implication from (a) to (b). Let $A \underset{*}{\leqslant} B$ and assume that $P, Q \in B(H)$ are idempotent operators such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$, $\operatorname{Ker} Q = \operatorname{Ker} A$, $PA = PB$, $AQ = BQ$, $P = P^*$, and $Q = Q^*$. Let $\operatorname{Im} Q = H_1$ and $\operatorname{Im} P = H_2$. Clearly H_1 and H_2 are closed subspaces of H hence $H = H_1 \oplus H_1^{\perp} = H_2 \oplus H_2^{\perp}$. Since P and Q are selfadjoint, we have $\operatorname{Ker} Q = H_1^{\perp}$ and $\operatorname{Ker} P = H_2^{\perp}$. Also since $\operatorname{Im} P = \overline{\operatorname{Im} A}$ and $\operatorname{Ker} Q = \operatorname{Ker} A$, we may conclude that the operator $A : H_1 \oplus H_1^{\perp} \to H_2 \oplus H_2^{\perp}$ has the following matrix representation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here $A_1 : H_1 \to H_2$ is an injective operator with $\overline{\operatorname{Im} A_1} = H_2$. From $(B-A)Q = 0$ it follows that

$$B = \begin{bmatrix} A_1 & B_2 \\ 0 & B_1 \end{bmatrix},$$

and by $P(B-A) = 0$ we have $\operatorname{Im}(B-A) \subseteq \operatorname{Ker} P$ and hence $B_2 = 0$.

Second, let us prove that (*b*) implies (*a*). Assume (*b*) holds. Then there exist selfadjoint idempotents $P, Q \in B(H)$ such that

$$\operatorname{Im} Q = H_1, \operatorname{Ker} Q = H_1^{\perp}, \operatorname{Im} P = H_2, \operatorname{Ker} P = H_2^{\perp}.$$

So, $\overline{\operatorname{Im} A} = H_2 = \operatorname{Im} P$ and $\operatorname{Ker} A = H_1^{\perp} = \operatorname{Ker} Q$. Also $PA = A = PB$ and $AQ = A = BQ$. The statements (*a*) and (*b*) are therefore equivalent.

Let us again assume that (*b*) holds and let us prove that it implies (*c*). Since $\operatorname{Im} A \subseteq H_2$ and $\operatorname{Im}(B-A) \subseteq H_2^{\perp}$, we may conclude that $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(B-A)}$. Thus, since (*a*) and (*b*) are equivalent, it remains to show that $A \underset{*}{\leqslant} B$ implies $A^* \underset{*}{\leqslant} B^*$. Let $A \underset{*}{\leqslant} B$ and let P, Q be as in Definition 2. Then $\operatorname{Ker} P^* = \operatorname{Ker} P = (\operatorname{Im} P)^{\perp} = \overline{\operatorname{Im} A}^{\perp} = \operatorname{Ker} A^*$ and $\operatorname{Im} Q^* = \operatorname{Im} Q = (\operatorname{Ker} Q)^{\perp} = (\operatorname{Ker} A)^{\perp} = \overline{\operatorname{Im} A^*}$. From $PA = PB$ we may conclude that $A^*P^* = B^*P^*$ and from $AQ = BQ$ we obtain $Q^*A^* = Q^*B^*$. It follows that $A^* \underset{*}{\leqslant} B^*$.

To conclude the proof we will show that (*c*) implies (*a*). There exists a selfadjoint idempotent $P \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$ and $\operatorname{Ker} P = \overline{\operatorname{Im} A}^{\perp}$. Since $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(B-A)}$, we have $\operatorname{Im}(B-A) \subseteq \overline{\operatorname{Im}(B-A)} \subseteq \operatorname{Ker} P$ and therefore $P(B-A) = 0$. So, $PA = PB$. Similarly we may find a selfadjoint idempotent $Q^* \in B(H)$ such that $\operatorname{Im} Q^* = \overline{\operatorname{Im} A^*}$ and $Q^*A^* = Q^*B^*$. So, $\operatorname{Ker} Q = \overline{\operatorname{Im} A^*}^{\perp} = \operatorname{Ker} A$ and $AQ = BQ$. It follows that $A \underset{*}{\leqslant} B$. $\square$

While proving Lemma 3 we showed the following corollary.

Corollary. *For all* $A, B \in B(H)$ *we have* $A \underset{*}{\leqslant} B$ *if and only if* $A^* \underset{*}{\leqslant} B^*$.

We will now prove that $\underset{*}{\leqslant}$ is indeed a partial order.

Theorem 4. *Relation* $\underset{*}{\leqslant}$, *defined by Definition 2, is a partial order*.

Proof. Let $A \in B(H)$. Since there exist selfadjoint idempotents $P, Q \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$ and $\operatorname{Ker} Q = \operatorname{Ker} A$, it trivially follows that $A \underset{*}{\leqslant} A$. Let now $A, B \in B(H)$ such that $A \underset{*}{\leqslant} B$ and $B \underset{*}{\leqslant} A$. By the statement (*b*) in Lemma 3 we have $\operatorname{Im} A \subseteq \operatorname{Im} B$ and $\operatorname{Im} B \subseteq \operatorname{Im} A$ and hence $\operatorname{Im} A = \operatorname{Im} B$. We may assume that A and B have the same matrix representation as in the statement (*b*) of Lemma 3. If $B_1 \neq 0$, then $\operatorname{Im} A \neq \operatorname{Im} B$, a contradiction. We conclude that $A = B$.

Let $A, B, C \in B(H)$ and assume that $A \underset{*}{\leqslant} B$ and $B \underset{*}{\leqslant} C$. Corollary yields that then $A^* \underset{*}{\leqslant} B^*$ and $B^* \underset{*}{\leqslant} C^*$. So it is enough to prove that $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(C-A)}$. First we observe $\overline{\operatorname{Im} A} \subseteq \overline{\operatorname{Im} B} \subseteq \overline{\operatorname{Im} C}$. Since $\overline{\operatorname{Im} B} \perp \overline{\operatorname{Im}(C-B)}$ we may conclude that $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(C-B)}$. Also

$$\operatorname{Im}(C-A) \subseteq \operatorname{Im}(C-B) + \operatorname{Im}(B-A) \subseteq \overline{\operatorname{Im}(C-B)} + \overline{\operatorname{Im}(B-A)}$$

Since $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(B-A)}$ and $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(C-B)}$, it follows that $\overline{\operatorname{Im} A} \perp \operatorname{Im}(C-A)$. Using the fact that the inner product is continuous we may conclude that $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(C-A)}$ and hence $A \underset{*}{\leqslant} C$. $\square$

We proved that the star order $\underset{*}{\leqslant}$ which was introduced in Definition 2 is a partial order. Now, let us justify the notation "the star order" to the partial order from Definition 2.

Theorem 5. *Let $\underset{*}{\leqslant}$ be defined by Definition 2. Then $A \underset{*}{\leqslant} B$ if and only if $A^*A = A^*B$ and $AA^* = BA^*$ for every $A, B \in B(H)$.*

Proof. Let us first assume that there exist idempotent operators $P, Q \in B(H)$ such that $\operatorname{Im} P = \overline{\operatorname{Im} A}$, $\operatorname{Ker} A = \operatorname{Ker} Q$, $PA = PB$, $AQ = BQ$, $P = P^*$ and $Q = Q^*$. From Lemma 3 we have $\overline{\operatorname{Im} A} \perp \overline{\operatorname{Im}(B-A)}$. Then $\langle (B-A)x, Ax\rangle = 0$ and therefore $\langle A^*(B-A)x, x\rangle = 0$, $x \in H$. It follows that $A^*A = A^*B$. Since by Lemma 3 also $\overline{\operatorname{Im} A^*} \perp \overline{\operatorname{Im}(B^*-A^*)}$, we obtain $AA^* = BA^*$.

Assume now that for $A, B \in B(H)$ we have $A^*A = A^*B$ and $AA^* = BA^*$. We will prove that there exists a selfadjoint idempotent P such that $PA = PB$ and $\operatorname{Im} P = \overline{\operatorname{Im} A}$. Recall that there exists a unique partial isometry W such that $A = \sqrt{AA^*}W$ is the polar decomposition of A with $\operatorname{Im} W = \overline{\operatorname{Im} \sqrt{AA^*}}$ (see for example [9, p. 75]). The equation $A^*A = A^*B$ yields $W^*\sqrt{AA^*}A = W^*\sqrt{AA^*}B$ and hence

$$WW^*\sqrt{AA^*}A = WW^*\sqrt{AA^*}B.$$

Since W is a partial isometry, WW^* is a selfadjoint idempotent with $\operatorname{Im} W = \operatorname{Im} WW^*$ (see for example [1, p. 244]). Recall that $\operatorname{Im} W = \overline{\operatorname{Im} \sqrt{AA^*}}$, so $\operatorname{Im} WW^* = \operatorname{Im} W = \overline{\operatorname{Im} \sqrt{AA^*}}$, and since WW^* is an identity on a subspace $\overline{\operatorname{Im} \sqrt{AA^*}}$ we obtain that

$$\sqrt{AA^*}A = \sqrt{AA^*}B. \tag{4}$$

Let $H_1 = \overline{\operatorname{Im} \sqrt{AA^*}}$. Since $\sqrt{AA^*}$ is a selfadjoint operator, it follows that $H_1^\perp = \operatorname{Ker} \sqrt{AA^*}$. Also, H_1 is a closed subspace of H therefore $H = H_1 \oplus H_1^\perp$ and there exists a selfadjoint idempotent $\widetilde{P} \in B(H)$ such that $\operatorname{Im} \widetilde{P} = H_1$ and $\operatorname{Ker} \widetilde{P} = H_1^\perp$. We will now prove that $\widetilde{P}A = \widetilde{P}B$. Since $\operatorname{Im} \widetilde{P} = \overline{\operatorname{Im} \sqrt{AA^*}}$ and $\operatorname{Ker} \widetilde{P} = \operatorname{Ker} \sqrt{AA^*}$, the operator $\sqrt{AA^*} : H_1 \oplus H_1^\perp \to H_1 \oplus H_1^\perp$ has the following matrix representation

$$\sqrt{AA^*} = \begin{bmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{bmatrix}.$$

Here $\widetilde{A} : H_1 \to H_1$ is an injective operator with $\overline{\operatorname{Im} \widetilde{A}} = H_1$. For $A, B : H_1 \oplus H_1^\perp \to H_1 \oplus H_1^\perp$ let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

From Eq. 4) it follows

$$\begin{bmatrix} \widetilde{A}A_1 & \widetilde{A}A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{A}B_1 & \widetilde{A}B_2 \\ 0 & 0 \end{bmatrix},$$

hence $\widetilde{A}A_1 = \widetilde{A}B_1$ and $\widetilde{A}A_2 = \widetilde{A}B_2$. Since $\widetilde{A}$ is injective, it follows $A_1 = B_1$ and $A_2 = B_2$.
The selfadjoint idempotent $\widetilde{P} : H_1 \oplus H_1^\perp \to H_1 \oplus H_1^\perp$ has the following matrix representation

$$\widetilde{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where $I : H_1 \to H_1$ is the identity operator. It follows that

$$\widetilde{P}A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{P}B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}.$$

Since $A_1 = B_1$ and $A_2 = B_2$ we may conclude that

$$\widetilde{P}A = \widetilde{P}B. \tag{5}$$

Recall that $\mathrm{Im}\,\widetilde{P} = \overline{\mathrm{Im}\sqrt{AA^*}}$. Since $A = \sqrt{AA^*}W$, it follows that $\mathrm{Im}\,A \subseteq \mathrm{Im}\,\sqrt{AA^*}$ and hence $\overline{\mathrm{Im}\,A} \subseteq \overline{\mathrm{Im}\,\sqrt{AA^*}}$. There exists a selfadjoint idempotent $P \in B(H)$ such that $\mathrm{Im}P = \overline{\mathrm{Im}\,A}$. So $\mathrm{Im}\,P \subseteq \mathrm{Im}\,\widetilde{P}$ which yields $P\widetilde{P} = \widetilde{P}P = P$. From Eq. 5) it follows $P\widetilde{P}A = P\widetilde{P}B$. We conclude that $PA = PB$.

The existence of a selfadjoint idempotent $Q \in B(H)$ such that $\mathrm{Ker}A = \mathrm{Ker}\,Q$ and $AQ = BQ$ can be proved in a similar way. Here we use the polar decomposition $A = V\sqrt{A^*A}$ where V is a unique partial isometry with $\mathrm{Ker}\,\sqrt{A^*A} = \mathrm{Ker}\,V$, and we define Q to be selfadjoint with $\mathrm{Im}\,Q = \overline{\mathrm{Im}A^*}$. $\square$

3. The generalized concept of order relations on $B(H)$

Mitra in [6] showed in his unified theory of the matrix partial orders that for any definition of a generalized inverse we may define an order on M_n.

Definition 6. For every $A \in M_n$ let $\mathcal{G}'(A)$ denote the set of all generalized inverses of A. An order $\leqslant_{\mathcal{G}'}$ on M_n is defined by

$$A \leqslant_{\mathcal{G}'} B \text{ if and only if } AG = BG \quad \text{and} \quad GA = GB$$

for some $G \in \mathcal{G}'(A)$.

Šemrl proposed in [8] a similar unified approach on $B(H)$. Let $I(H) \subseteq B(H)$ be the set of all bounded idempotent operators on a Hilbert space H. We start with a pair of maps $\phi, \psi : B(H) \to \mathcal{P}(I(H))$ which map every operator $A \in B(H)$ to two specified sets of idempotents $\phi(A), \psi(A) \subseteq I(H)$. Let $\phi(A)$ be the set of idempotents $Q \in I(H)$ that satisfy either one, or two, or even all three of the following conditions

$$\mathrm{Ker}\ A \subseteq \mathrm{Ker}\ Q, \tag{6}$$

$$\mathrm{Ker}\ Q \subseteq \mathrm{Ker}\ A, \tag{7}$$

$$Q = Q^*. \tag{8}$$

Choose $\psi(A)$ to be the set of idempotents $P \in I(H)$ that satisfy either one, or two, or even all three of the following conditions

$$\mathrm{Im}\ A \subseteq \mathrm{Im}\ P, \tag{9}$$

$$\mathrm{Im}\ P \subseteq \overline{\mathrm{Im}\ A}, \tag{10}$$

$$P = P^*. \tag{11}$$

Definition 7. For every $A \in B(H)$ let $\phi(A)$ and $\psi(A)$ be the sets of idempotents defined before. An order $\leqslant_{\phi,\psi}$ on $B(H)$ is defined by $A \leqslant_{\phi,\psi} B$ if and only if $AQ = BQ$ and $PA = PB$ for some $Q \in \phi(A)$ and some $P \in \psi(A)$.

If conditions (6), (7), (9), and (10) are satisfied, then clearly $\leqslant_{\phi,\psi}$ is the minus partial order on $B(H)$. Similarly, if all six conditions (6)–(11) are satisfied, then $\leqslant_{\phi,\psi}$ is obviously the star order on $B(H)$. Actually, the following proposition can be proved.

Proposition. *Let $A, B \in B(H)$ and assume only conditions (7), (8), (9), and (11) are satisfied. Then $A \underset{*}{\leqslant} B$ if and only if $A \leqslant_{\phi,\psi} B$, i.e., $AQ = BQ$ and $PA = PB$ for some idempotent $Q \in B(H)$ satisfying* $\operatorname{Ker} Q \subseteq \operatorname{Ker} A$, $Q = Q^*$, *and for some idempotent $P \in B(H)$ satisfying* $\operatorname{Im} A \subseteq \operatorname{Im} P$, $P = P^*$.

Proof. Let $A \leqslant_{\phi,\psi} B$ and suppose P and Q are as above. Let $P_0 \in B(H)$ be the selfadjoint idempotent operator defined by $\operatorname{Im} P_0 = \overline{\operatorname{Im} A}$. Similarly, let $Q_0 \in B(H)$ be the selfadjoint idempotent such that $\operatorname{Ker} Q_0 = \operatorname{Ker} A$. It follows that $\operatorname{Im} P_0 \subseteq \operatorname{Im} P$ and $\operatorname{Im} Q_0 \subseteq \operatorname{Im} Q$. So, $P_0P = PP_0 = P_0$ and $Q_0Q = QQ_0 = Q_0$. Since $PA = PB$, we may conclude that $P_0A = P_0B$, and similarly $AQ_0 = BQ_0$. It follows that $A \underset{*}{\leqslant} B$. Clearly, if $A \underset{*}{\leqslant} B$, then $A \leqslant_{\phi,\psi} B$. So, such order $\leqslant_{\phi,\psi}$ is equivalent to the star order $\underset{*}{\leqslant}$. $\square$

Of course we can not expect to get a partial order if we take any subset of the six conditions. Obviously any order $\leqslant_{\phi,\psi}$ is reflexive, however it is not necessarily transitive or even antisymmetric.

Example 8. Let H_1 be a Hilbert space, $H = H_1 \oplus H_1 \oplus H_1$, $A, B \in B(H)$, and $I \in B(H_1)$ identity operator. Suppose $A \leqslant_{\phi,\psi} B$ if and only if $AQ = BQ$ and $PA = PB$ for some $Q \in \phi(A)$ satisfying (7) and $P \in \psi(A)$ satisfying (10). Let

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & 0 & -I \\ 0 & 2I & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Take $Q_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $AQ_1 = BQ_1$ and $\operatorname{Ker} Q_1 = \operatorname{Ker} A$. The condition (10) may be satisfied by using $P_1 = 0$, hence we have $A \leqslant_{\phi,\psi} B$.

Let now $Q_2 = \begin{bmatrix} I & 0 & -I \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then Q_2 is an idempotent, $\operatorname{Ker} Q_2 = \operatorname{Ker} B$ and $BQ_2 = B = AQ_2$. Again (10) is fulfilled if we take $P_2 = 0$. It follows that $B \leqslant_{\phi,\psi} A$. So $\leqslant_{\phi,\psi}$ is not antisymmetric since $A \neq B$.

If we add condition (8) to the order from the previous example, we obtain an example of an order $\leqslant_{\phi,\psi}$ that is antisymmetric and reflexive, but it is not transitive.

Example 9. Let $B(H)$ be as in the previous example and let $A, B \in B(H)$. Suppose that $A \leqslant_{\phi,\psi} B$ if and only if $AQ = BQ$ and $PA = PB$ for some $Q \in \phi(A)$ satisfying (7), (8) and $P \in \psi(A)$ satisfying (10).

Let $A \leqslant_{\phi,\psi} B$ and $B \leqslant_{\phi,\psi} A$ for some $A, B \in B(H)$. On the one hand there exist idempotents Q_1 and P_1 such that $AQ_1 = BQ_1$ and $P_1A = P_1B$ satisfying $\operatorname{Ker} Q_1 \subseteq \operatorname{Ker} A$, $Q_1 = Q_1^*$, $\operatorname{Im} P_1 \subseteq \overline{\operatorname{Im} A}$. On the other hand there exist idempotents Q_2 and P_2 such that $AQ_2 = BQ_2$ and $P_2A = P_2B$ satisfying $\operatorname{Ker} Q_2 \subseteq \operatorname{Ker} B$, $Q_2 = Q_2^*$, $\operatorname{Im} P_2 \subseteq \overline{\operatorname{Im} B}$. Note that we may satisfy the condition (10) and equations $P_iA = P_iB$, $i \in \{1, 2\}$, by taking $P_1 = P_2 = 0$.

The spaces $\operatorname{Ker} Q_1$ and $\operatorname{Ker} Q_2$ are closed hence $\operatorname{Ker} Q_1 \cap \operatorname{Ker} Q_2$ is also a closed space. It follows that

$$H = (\operatorname{Ker} Q_1 \cap \operatorname{Ker} Q_2) \oplus (\operatorname{Ker} Q_1 \cap \operatorname{Ker} Q_2)^{\perp}.$$

There exists a selfadjoint idempotent Q such that

$$\operatorname{Ker} Q = \operatorname{Ker} Q_1 \cap \operatorname{Ker} Q_2 \quad \text{and} \quad \operatorname{Im} Q = (\operatorname{Ker} Q_1 \cap \operatorname{Ker} Q_2)^{\perp}.$$

Hence $\operatorname{Ker} Q \subseteq \operatorname{Ker} A$ and $\operatorname{Ker} Q \subseteq \operatorname{Ker} B$. So $Ax = Bx = 0$ for every $x \in \operatorname{Ker} Q$. Subspaces $\operatorname{Ker} Q_1 \subseteq H$ and $\operatorname{Ker} Q_2 \subseteq H$ are closed, therefore $(\operatorname{Ker} Q_1 \cap \operatorname{Ker} Q_2)^{\perp} = (\operatorname{Ker} Q_1)^{\perp} + (\operatorname{Ker} Q_2)^{\perp}$. It follows that

$$\operatorname{Im} Q = (\operatorname{Ker} Q_1)^{\perp} + (\operatorname{Ker} Q_2)^{\perp} = \operatorname{Im} Q_1 + \operatorname{Im} Q_2.$$

Let $y \in \operatorname{Im} Q$. Then there exist $u \in \operatorname{Im} Q_1$ and $v \in \operatorname{Im} Q_2$ such that $y = u + v$. For $i \in \{1, 2\}$ we have $AQ_i = BQ_i$, so $Ax = Bx$ for every $x \in \operatorname{Im} Q_1 \cup \operatorname{Im} Q_2$. It follows that $Ay = Au + Av = Bu + Bv = By$, $y \in \operatorname{Im} Q$. Since $H = \operatorname{Ker} Q \oplus \operatorname{Im} Q$ we obtain that $A = B$.

Lastly, let us show that this order is not transitive. Let

$$A = \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 & I \\ 0 & 2I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 & I \\ 0 & 2I & 0 \\ I & 0 & -I \end{bmatrix}$$

and $Q_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since $AQ_1 = BQ_1$ and $\operatorname{Ker} Q_1 = \operatorname{Ker} A$, we have $A \leqslant_{\phi,\psi} B$. Let

$$Q_2 = \begin{bmatrix} \frac{1}{2}I & 0 & \frac{1}{2}I \\ 0 & I & 0 \\ \frac{1}{2}I & 0 & \frac{1}{2}I \end{bmatrix}.$$

Then $Q_2^2 = Q_2, Q_2^* = Q_2$, and $\operatorname{Ker} Q_2 = \operatorname{Ker} B$. Also, $BQ_2 = CQ_2$, hence $B \leqslant_{\phi,\psi} C$. Take a selfadjoint idempotent

$$Q_3 = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}.$$

We have

$$AQ_3 = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 2P_{21} & 2P_{22} & 2P_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$CQ_3 = \begin{bmatrix} P_{11} + P_{31} & P_{12} + P_{32} & P_{13} + P_{33} \\ 2P_{21} & 2P_{22} & 2P_{23} \\ P_{11} - P_{31} & P_{12} - P_{32} & P_{13} - P_{33} \end{bmatrix}.$$

Suppose that $AQ_3 = CQ_3$. It follows that $P_{31} = P_{32} = P_{33} = 0$ and hence $P_{11} = P_{12} = P_{13} = 0$. Since $Q_3^* = Q_3$, we have $P_{21} = P_{23} = 0$. So

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The fact that $\operatorname{Ker} Q_3 \nsubseteq \operatorname{Ker} A$ yields that $A \not\leqslant_{\phi,\psi} C$.

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